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Reduced systems of (2, 2) pseudo-Euclidean noncommutative self-dual Yang–Mills theories

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Abstract
Self-dual Yang–Mills equations on noncommutative spaces associated with pseudo-Euclidean space of signature (2, 2) are shown to be related via dimensional reductions to noncommutative formulations of Toda equations, of generalized nonlinear Schrödinger (NS) equations, of the super-Korteweg–de Vries (super-KdV) as well as of the matrix KdV equations. The noncommutative extensions of their linear systems and bicomplexes associated with conserved quantities are discussed as well. A q-plane version of the KdV equation with linear system is also shown.

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1. Introduction

Noncommutative geometry has been involved in noncommutative versions of gauge theories in relation to strings (for example [1]), and specifically in noncommutative self-dual Yang–Mills systems [2] and gauge theories of gravity [3]. A number of papers have been devoted to the study of field theories on noncommutative spaces (for instance [4, 5]), and some properties have been probed as well as solutions investigated (for examples [6–8], and a review [5]). Classical integrable models have also been extended to noncommutative spaces (for instance, see [6–15]), and as examples, noncommutative versions of the Toda, nonlinear σ-model, (cubic) nonlinear Schrödinger (NS) and Korteweg–de Vries (KdV) equations have been formulated on such spaces, leading to 'deformed' versions of these systems. With the help of bicomplexes, an infinite set of conserved quantities has been found, which suggests (complete) integrability of these modified systems [10, 11, 15], and solutions, such as 'deformed' solitons, were presented in certain cases [6, 11, 16–18]. Following results on the deformed ADHM construction [19] and its twistor interpretation [20], a formulation on (anti-) self-dual Yang–Mills (with the abbreviation: (Anti-)SDYM, below) equations has been shown, along with twistor and corresponding integrability properties preserved in this setting [21]. Deformed (Anti-)SDYM equations are simply obtained by substituting the usual product by the Moyal product in the classical version of these equations. However, different formulations can be derived by using diverse products on different noncommutative space structures, either the canonical, the Lie algebra or quantum space structures [4]. The dimensional reductions of the Moyal product (canonical structure) formulation of the (Anti-)SDYM equations to a (noncommutative) chiral field model and Hitchin equations are discussed in [6, 21], as well as integrability properties inherited from the (Anti-)SDYM equations, and (soliton) solutions.

In this short paper SDYM equations on noncommutative versions of pseudo-Euclidean spaces endowed with a metric of signature (2, 2), denoted $\mathbb{E}^{(2,2)}$, are dimensionally reduced with the help of certain sets of conditions, or constraints, on the gauge fields. The derivations shown of the noncommutative systems from the SDYM equations also rely on the same approach and methods used for ordinary gauge fields, which could here be valued in an enveloping algebra of a Lie algebra [22]. However, one notes that not all commutative reductions of (Anti-)SDYM can be extended to noncommutative ones. A reduction to a noncommutative KdV equation on a q-plane is also found, along with a corresponding linear system. Once Lax equations or a linear system have been exhibited, the next step would be to attempt applying the dressing method in order to derive solutions (for example [23]). This work is planned as a follow-up.

2. Bicomplexes and (noncommutative) linear systems

Here, bicomplexes and linear systems are rapidly mentioned in view of the relation of the former notion with conservation laws. Definitions and applications of bicomplexes can be found for example in the following [25–27]. For our purposes, let us use the following definition below (see for example [27]). A bicomplex corresponds to a linear space over $\mathbb{R}$ or $\mathbb{C}$, here denoted V, endowed with a grading over the non-negative integers, i.e. $V = \oplus_{i\geqslant 0} V^i$, and two (linear) maps (operators) d and δ between successive spaces V^i and V^{i+1}, in other words, $d : V^i \to V^{i+1}$, and $\delta : V^i \to V^{i+1}$, such that $d^2\cdot = 0$, $\delta^2\cdot = 0$, $(\delta d + d\delta)\cdot = 0$, where $\cdot$ stands for an element of V.

On the four-dimensional pseudo-Euclidean space $\mathbb{E}^{(2,2)}$ with diagonal metric $(+, +, -, -)$, one finds the following set of linear equations [28] for the SDYM equations on $\mathbb{E}^{(2,2)}$, with a change to null variables [24, 28]: $t = \frac{1}{\sqrt{2}}(x^2 - x^4)$, $y = \frac{1}{\sqrt{2}}(x^1 - x^3)$, $u = \frac{1}{\sqrt{2}}(x^2 + x^4)$, $z = \frac{1}{\sqrt{2}}(x^1 + x^3)$:

$$\begin{aligned} &(D_z + \omega D_u)\Psi(x, \omega, \bar{\omega}) = 0 \\ &(D_t - \omega D_y)\Psi(x, \omega, \bar{\omega}) = 0 \\ &\partial_{\bar{\omega}}\Psi(x, \omega, \bar{\omega}) = 0 \end{aligned} \tag{1}$$

where the parameter $\omega = \mathrm{i}\frac{(1-\lambda)}{(1+\lambda)}$, $\lambda \in$ (a sheet of hyperboloid $\mathbb{H}^2$) and $D_t = \frac{1}{\sqrt{2}}(D_2 - D_4)$, $D_u = \frac{1}{\sqrt{2}}(D_2 + D_4)$, $D_y = \frac{1}{\sqrt{2}}(D_1 - D_3)$, $D_z = \frac{1}{\sqrt{2}}(D_1 + D_3)$, with $D_\mu = \partial_\mu + A_\mu$, $\mu = 1, 2, 3, 4$.

One can build a bicomplex based on the previous type of linear systems (1) with parameter ω:

$$\mathcal{D}_1\Psi = \left[\mathcal{O}_1 + \omega\mathcal{O}_1^\omega\right]\Psi = 0 \qquad \mathcal{D}_2\Psi = \left[\mathcal{O}_2 + \omega\mathcal{O}_2^\omega\right]\Psi = 0 \tag{2}$$

as the set of two operators d and δ on $\Psi(\omega) : \mathbb{R}^{(2,2)} \to \mathbb{C}^n, \in V^0$ [9, 10]:

$$d\Psi = \mathcal{O}_1\Psi\xi_1 + \mathcal{O}_2\Psi\xi_2 \qquad \delta\Psi = \mathcal{O}_1^\omega\Psi\xi_1 + \mathcal{O}_2^\omega\Psi\xi_2 \tag{3}$$

or, in short, $d\Psi + \omega\delta\Psi = 0$ (which resembles the 'linear equation' formulation of [10, 11]), where $\xi_1, \xi_2 \in \Lambda^1$. Then the conditions for these operators to form a bicomplex: $d^2 = 0$,

$\delta^2 = 0$, $d\delta + \delta d = 0$, correspond exactly to the compatibility or integrability conditions of the linear system (2), and provide the SDYM equations. Noncommutative structures are recalled in [4]. Here, the canonical structure will be used:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \qquad \text{where} \quad \theta^{\mu\nu} \in \mathbb{C} \tag{4}$$

One notes that the quantum space structure with $n = 2$ includes the quantum (or Manin) plane [30, 31] as a possible case: $\hat{y}\hat{x} = q\hat{x}\hat{y}$, $q \in \mathbb{C}$. Moreover, the associative product on the function spaces on $\mathbb{R}^{(2,2)}$ is for the canonical structure replaced by the associative but noncommutative Moyal (–Weyl) product [32], denoted by $*$, for two classical functions f and g:

$$(f * g)(x) = \exp\left[\sum_{\mu,\nu=1}^{4} \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{\tilde{x}^\nu}\right] f(x^\lambda)g(\tilde{x}^\sigma)\bigg|_{x^\mu=\tilde{x}^\mu}. \tag{5}$$

For the q-plane, one recalls that the product is based on the normal ordering of the operator $\hat{x}$, $\hat{y}$, where the $\hat{x}$-operators are left-end sided, and the $\hat{y}$-operators are right-end sided [4]:

$$: f(\hat{x}, \hat{y}) :: g(\hat{x}, \hat{y}) :=: (f \diamond g)(\hat{x}, \hat{y}) : \tag{6}$$

where $\diamond$ denotes the product of two classical functions in the x, y variables:

$$(f \diamond g)(x, y) = q^{(\tilde{x}\partial_{\tilde{x}}y\partial_y)} f(x, y)g(\tilde{x}, \tilde{y})|_{\tilde{x}=x,\tilde{y}=y}. \tag{7}$$

An extension of such $\diamond$ products could be built for higher dimensional quantum spaces and different orderings could also be implemented in the same fashion. Derivations can also be defined on quantum spaces, and in particular for the q-plane, one can introduce [31] algebra automorphisms σ_x and σ_y:

$$\sigma_x(\hat{x}) = q\hat{x} \quad \sigma_x(\hat{y}) = \hat{y} \qquad \text{and} \qquad \sigma_y(\hat{x}) = \hat{x} \quad \sigma_y(\hat{y}) = q\hat{y} \tag{8}$$

and q-derivatives, denoted ∂_x^q and ∂_y^q, which are endomorphisms on the q-plane, corresponding to a $\left(\sigma_x^{-1}\sigma_y, \sigma_x\right)$-derivation and a $\left(\sigma_y, \sigma_x\sigma_y^{-1}\right)$-derivation, respectively. Explicitly, one has on the q-plane basis elements $(x^m y^n)$, the following q-derivatives, extensions of the usual x and y derivatives [31]:

$$\partial_x^q \hat{x}^m\hat{y}^n = [m]\hat{x}^{m-1}\hat{y}^n \qquad \text{and} \qquad \partial_y^q \hat{x}^m\hat{y}^n = [n]\hat{x}^m\hat{y}^{n-1} \tag{9}$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$, with $q^2 \neq 1$. Note that other q-derivations have been defined [33].

For example, if f is an element of $\mathcal{A}$, which denotes the algebra of formal power series in $\hat{x}$ and $\hat{y}$ modulo the relation $\hat{y}\hat{x} = q\hat{x}\hat{y}$, a generalization of the two-dimensional wave equation can be written in terms of the above derivations on the q-plane:

$$\left[\left(\partial_x^q\right)^2 - \left(\partial_y^q\right)^2\right] f(\hat{x}, \hat{y}) = 0. \tag{10}$$

The well known general (wave) solutions: $F(\hat{x} + \hat{y}) + G(\hat{x} - \hat{y})$ are not necessarily valid, but solutions can be expressed as q-series.

3. 2D reductions for a canonical structure

A version of the (Anti-)SDYM equations on $\mathbb{E}^{(2,2)}$ endowed with the canonical structure can be obtained by using the $*$-product instead of the usual commutative product of two functions [21, 22]. Thus

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \overset{*}{,} A_\nu] \tag{11}$$

stands for the field strength of the gauge field components A_μ. The (Anti-)SDYM equations are then invariant under the (infinitesimal) gauge transformations:

$$\delta_g A_\mu = \partial_\mu \Lambda + [A_\mu \overset{*}{,} \Lambda] \tag{12}$$

with Λ having values in the gauge algebra. It is noted that ∂_μ is still a 'derivation' for the canonical structure (4). In the following, slight modifications of known reductions of the (commutative) SDYM equations to (classical) integrable systems will be used to derive noncommutative generalizations of the same integrable models, using 'translational' reductions of the SDYM equations, helped with constraints on the gauge fields (with values in the enveloping algebra of a Lie algebra). Reductions [34] involving gauge fields with values in infinite-dimensional algebras have already been performed in order to derive various integrable systems [29].

3.1. *Toda field equations*

From the linear system (1) given in terms of the z, u, t, y variables, the noncommutative version of the Toda equations, as given in [9], is derived by requiring translation symmetries along the y and z variables, with $\partial_u = \partial_T - \partial_x$ and $\partial_t = \partial_T + \partial_x$, and imposing the following constraints on the fields components:

$$A_z = -L \qquad A_t = M \qquad A_y = -(S - I) \qquad A_u = 0 \tag{13}$$

where [9]

$$\begin{aligned} &S = \sum_{i=1}^{(n-1)} E_{i,i+1} \qquad L = G^{-1} * S^T * G \\ &M = G^{-1} * (G_t + G_x) \qquad G = \sum_{i=1}^{n} G_i E_{i,i} \end{aligned} \tag{14}$$

with

$$[E_{i,j}]^k_l = \delta^k_i \delta_{j,l} \qquad G_i^{-1} * G_i = 1 \qquad i = 1, \ldots, n. \tag{15}$$

Solutions have been given using the notion of quasideterminant in [16, 17] for non-Abelian cases.

3.2. *Generalized NS equations*

Guided by [29, 35], noncommutative NS equations can be derived from the SDYM equations on $\mathbb{E}^{(2,2)}$ using translational invariance along the coordinates y and $t - u$, with the 'ansatz'

$$\begin{aligned} &A_u = 0 \qquad && A_t = \begin{bmatrix} 0 & -q \\ r & 0 \end{bmatrix} \\ &A_y = -\kappa \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad && A_z = -\frac{1}{2\kappa} \begin{bmatrix} q * r & q_x \\ r_x & -r * q \end{bmatrix}. \end{aligned} \tag{16}$$

The residual linear system (1) takes the form

$$[\partial_x + (A_t - \omega A_y)] * \Psi = 0 \qquad [\partial_T + \omega \partial_x + A_z] * \Psi = 0 \qquad \partial_{\bar{\lambda}} \Psi = 0 \tag{17}$$

and then have the noncommutative generalized NS equations given below as compatibility equations:

$$2\kappa q_T = q_{xx} + 2q * r * q \qquad 2\kappa r_T = -(r_{xx} + 2r * q * r) \tag{18}$$

where $x = t + u$, $T = z$ and κ is a constant.

Equations (18) coincide with the equations obtained in [10] from an almost similar bicomplex, with $q = \bar{r}$ and $\kappa = \frac{i}{2}$. Let us add that conserved quantities for this noncommutative system would be derivable in a manner similar to the approach found in [10]. In [17], a one dependent variable $q = \bar{r}$ case is discussed.

3.3. Super matrix KdV equations

In a follow-up to the work of [35, 36] on the reduction of the (Anti-)SDYM equations to the (commutative) KdV equation, the (commutative) matrix KdV equations were obtained from the same original (Anti-)SDYM equations by imposing invariance under translations [28], and then using Lie superalgebra valued gauge fields, a supersymmetric version of the matrix KdV model was found [24].

The symmetries and ansatzes provided in [29, 35] have not allowed us to derive a noncommutative form of these equations via the same procedure used in noncommutative (Anti-)SDYM equations. Instead, the formulations of [24, 28, 36] have been found more suitable for this purpose.

Starting from the linear system (1) and requiring translational symmetries along the coordinates u and $y - z$, one derives the residual linear equations

$$[\partial_t + A_t + \omega(A_z - A_y) + \omega^2 A_u] * \Psi = 0 \qquad [\partial_x + A_z + \omega A_u] * \Psi = 0. \tag{19}$$

A noncommutative version of a supersymmetric (matrix) KdV equation can be produced from the linear systems (19) by inserting the following ansatz for the gauge field components into the compatibility equations:

$$\begin{aligned} &A_u = \begin{bmatrix} 0_n & 0_n & 0_n \\ -1_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix} && A_z = \begin{bmatrix} 0_n & 0_n & 0_n \\ U_n & 0_n & 0_n \\ 0_n & 0_n & \theta\phi_n \end{bmatrix} \\ &A_{z-y} = A_z - A_y = \begin{bmatrix} 0_n & 0_n & 0_n \\ 0_n & 0_n & \theta 1_n \\ \theta 1_n & 0_n & 0_n \end{bmatrix} && A_y = \begin{bmatrix} 0_n & 0_n & 0_n \\ U_n & 0_n & -\theta 1_n \\ -\theta 1_n & 0_n & \theta\phi_n \end{bmatrix} \\ &A_t = \begin{bmatrix} 0_n & 0_n & 0_n \\ a_{12} & 0_n & 0_n \\ 0_n & 0_n & \theta a_{33} \end{bmatrix} \end{aligned} \tag{20}$$

where

$$a_{12} = 3U_n * U_n + U_{n,xx} - \tfrac{3}{2}\phi_n * \phi_{n,x} + \tfrac{3}{2}\phi_{n,x} * \phi_n \tag{21}$$

$$a_{33} = \phi_{n,xx} + \tfrac{3}{2}\phi_n * U_n + \tfrac{3}{2}U_n * \phi_n. \tag{22}$$

The reduced SDYM equations using the A_μ fields with values in the Lie superalgebra $gl(2n/n) \otimes \mathcal{A}$, where $\mathcal{A}$ identifies the set of functions on noncommutative $\mathbb{R}^{(2,2)}$, θ is an odd Grassmann variable, U_n and ϕ_n are, respectively, $n \times n$ matrices with even and odd degree variables depending on x and t, have the form

$$\begin{aligned} U_{n,t} &= 3U_{n,x} * U_n + 3U_n * U_{n,x} + U_{n,xxx} - \tfrac{3}{2}\phi_n * \phi_{n,xx} + \tfrac{3}{2}\phi_{n,xx} * \phi_n \\ \phi_{n,t} &= \phi_{n,xxx} + \tfrac{3}{2}\phi_{n,x} * U_n + \tfrac{3}{2}\phi_n * U_{n,x} + \tfrac{3}{2}U_{n,x} * \phi_n + \tfrac{3}{2}U_n * \phi_{n,x}. \end{aligned} \tag{23}$$

It can be verified that these noncommutative equations are left invariant under the following supersymmetry transformations, induced by the odd Grassmann parameter ϵ:

$$\delta_\epsilon U_n = \epsilon\phi_{n,x} \quad \text{and} \quad \delta_\epsilon \phi_n = \epsilon U_n. \tag{24}$$

When the odd variables ϕ_n are made to vanish, one obtains the matrix KdV equations

$$U_{n,t} = 3(U_{n,x} * U_n + U_n * U_{n,x}) + U_{n,xxx} \tag{25}$$

(another linear system for this equation can be found in [28]) which leads for $n = 1$ to

$$U_t = 3(U_x * U + U * U_x) + U_{xxx} \tag{26}$$

originally presented in [11] as a noncommutative version of the KdV equation, following a different path. An infinite set of conserved densities can be derived using a noncommutative version of the transformation presented in [37]: $U = W + \lambda W_x + \lambda^2 W * W$ [11]. Results (without $*$-product) can be found in [16] using the notion of quasideterminant for the $n = 1, \phi_n = 0$ case. One can add that the formulation of [36] can also lead to the same noncommutative version of the KdV equations, which arise as the compatibility of the linear system (19).

4. 2D reductions to a quantum plane

In this section, a dimensional reduction of the SDYM equations written on a quantum space is used to derive a noncommutative formulation of the KdV equation on the q-plane. Similar reductions of Toda, generalized NS and super matrix KdV equations on the q-plane can be thought of, and since the steps are closely related to those of the previous section, let us focus on a q-plane KdV equation derivation. However, it is mentioned that the (Anti-)SDYM equations on quantum space are not necessarily invariant under the gauge transformations (12), and do not arise as the compatibility of the similarly transformed linear system from commutative to noncommutative by substituting as done before for the canonical structure, derivatives with q-derivatives and products with $\diamond$-products. Here the main objective is to gain a q-plane version (related deformed version) of certain equations. Instead, one could have started straight from the KdV equation, by using the above substitutions with respect to the derivatives and products directly. However, a linear system has been found for the reduced version presented below, and not the direct version.

First, one picks the simplest quantum space structure which accommodates a q-plane after dimensional reduction. One imagines that different quantum spaces structure in four dimensions could lead via reduction to systems on two-dimensional quantum spaces, such as a q-plane. For our purpose, the 'coordinates' $x = y + z, w = y - z, u$, and t of the linear system (1), obey the following quantum space structure equations:

$$\begin{array}{lll} tx = qxt & wx = xw & wt = tw \\ tu = ut & wx = xw & wu = uw. \end{array} \tag{27}$$

Then, dimensional reductions are imposed along the w and u directions to retrieve the following reduced system with q-derivations with respect to x and t [24]:

$$\begin{aligned} &\partial_x^q A_u + [A_y \overset{\diamond}{,} A_u] = 0 \\ &\partial_t^q A_u - \partial_x^q (A_z - A_y) + [A_z \overset{\diamond}{,} A_y] + [A_t \overset{\diamond}{,} A_u] = 0 \\ &\partial_t^q A_z - \partial_x^q A_t + [A_t \overset{\diamond}{,} A_z] = 0. \end{aligned} \tag{28}$$

As done previously, the gauge components are valued in $sl(2, \mathbb{C}) \otimes \mathcal{A}$, with A_u, A_z, A_y being of the same form as given in (20) for $n = 1$ with $\phi_n, \theta = 0$, and

$$A_t(t, x) = \begin{bmatrix} 0 & 0 \\ 3U \diamond U + \left(\partial_x^q\right)^2 U & 0 \end{bmatrix}. \tag{29}$$

Substitution of these fields in the above system (28) gives rise to an extension of the KdV equation on the q-plane:

$$\partial_t^q U = 3\big(\sigma_x^{-1}\sigma_t(U) \diamond \partial_x^q U + \big(\partial_x^q U\big)\sigma_x(U)\big) + (\partial_x^q)^3 U \tag{30}$$

which compares to previous noncommutative results with modifications on the derivations and product.

Solutions can be sought for $U(t, x)$, as formal series expansions:

$$: U(t, x) := \sum_{i,j=0}^{\infty} c_{ij} : x^i t^j : \tag{31}$$

where $c_{ij} \in \mathbb{C}$ for $i, j \geqslant 0$. For the (commutative) KdV equation, Taylor series expansions with respect to initial values have been studied in [38, 39]. In a similar manner, formal series could be written for $: U(t, x) :$ of the noncommutative KdV equation with respect to 'initial' conditions. Solutions of these deformed systems would still have to be analysed. These equations differ from the q-difference equations found for example in [40], but with solutions written in terms of ('Fourier type') series [41].

A linear system to this q-plane KdV equation can now be built with a reference to the previous section. From the following set of linear equations where t plays the role of y as a generator of the q-plane:

$$\mathfrak{D}_x \diamond f = \big(\partial_x^q + X\Lambda\big) \diamond f = 0 \tag{32}$$

$$\mathfrak{D}_t \diamond f = \big(\partial_t^q + T\Sigma\big) \diamond f = 0 \tag{33}$$

with $X, T \in gl(2, \mathbb{C}) \otimes \mathcal{A}$, Λ, Σ are endomorphisms to be specified, and f represents a 2×1 column vector with $\mathcal{A}$ valued components. The compatibility equation has the form

$$(\mathfrak{D}_x \diamond \mathfrak{D}_t - \mathfrak{D}_t \diamond \mathfrak{D}_x) \diamond f = 0 \tag{34}$$

which explicitly can be written as

$$\big[\partial_x^q(T) - \partial_t^q(X)\big] \diamond f + \sigma_t(X) \diamond T \diamond \sigma_x^{-1}\sigma_t\sigma_x^{-1}(f) - \sigma_x^{-1}\sigma_t(T) \diamond X \diamond \sigma_t\sigma_x^{-1}\sigma_x^{-1}(f) = 0 \tag{35}$$

where the endomorphisms $\Sigma = \sigma_x^{-1}$ and $\Lambda = \sigma_t\sigma_x^{-1}$ were chosen.

Now, selecting

$$X = \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} 0 & 0 \\ 3U \diamond U + \big(\partial_x^q\big)^2 U & 0 \end{bmatrix} \tag{36}$$

the compatibility condition above simply becomes

$$\big[\partial_x^q(T) - \partial_t^q(X)\big] \diamond f = 0 \tag{37}$$

which for all f, implies the KdV equation on the q-plane found already.

5. Summary/conclusion

This paper has shown a relation via reductions through translations or dimensional reductions between a noncommutative version of self-dual Yang–Mills equations and noncommutative formulations of diverse integrable systems: Toda, generalized NS and matrix KdV equations, as well as the super matrix KdV system. It could then be seen as an extension of results published in [24, 35] in the direction of noncommutative theories. For each of these noncommutative versions of integrable models, a corresponding noncommutative linear system has been exhibited, and a link to bicomplexes provided. Conserved densities could be obtainable in a similar fashion to the cases presented in [10, 11]. A q-plane KdV equation with linear system has also been introduced.

Various directions can then be followed for future development. One may want to further explore the set of solutions and properties of these noncommutative models (for example [42]), using for instance a dressing method attempt with the linear systems presented. The integrability, and reduced twistor interpretations could also be probed, as well as further reductions to other integrable equations, using varied constraints and symmetries. Moreover, equations on quantum spaces can be explored, and other formulations of noncommutative gauge theories could be examined in a similar manner through a reduction process.

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